NORTH MAHARASHTRA UNIVERSITY,

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Question Bank

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Class : S.Y. B. Sc.

Subject : Mathematics

Paper : MTH – 212 (A) Abstract Algebra

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Question Bank

Paper : MTH – 212 (A)

Abstract Algebra

Unit – I

1 : Questions of 2 marks

1) Define product of two permutations on n symbols. Explain it by an example on 5 symbols.

2) Define inverse of a permutation. If σ = $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 4 & 2 & 7 & 6 \end{pmatrix} \in S_7$ then find σ^{-1}

3) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 6 & 3 \end{pmatrix}$ and $\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 1 & 4 \end{pmatrix}$ $\in S_6$. Find (i) $\lambda\,\sigma$ (ii) σ^{-1}.

4) Let f $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 6 & 3 \end{pmatrix}$ and g $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix} \in$ S_6. Find (i) f g (ii) g^{-1}.

5) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix} \in S_5$. Find $\alpha^{-1}\,\beta^{-1}$.

6) Define i) a permutation ii) a symmetric group.

7) Define i) a cycle ii) a transposition.

8) Let $C_1 = (2\ 3\ 7)$, $C_2 = (1\ 4\ 3\ 2)$ be cycles in S_8. Find C_1C_2 and express it as product of transpositions.

9) For any transposition $(a\ b) \in S_n$, prove that $(a\ b) = (a\ b)^{-1}$.

10) Prove that every cycle can be written as product of transpositions.

11) Define disjoint cycles. Are (1 4 7) , (4 3 2) disjoint cycles in S_8?

12) Write down all permutations on 3 symbols {1, 2, 3}.

13) Define an even permutation. Is $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$ an even permutation?

14) Define an odd permutation. Is $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 7 & 5 & 4 & 6 \end{pmatrix}$ an odd permutation?

15) Prove that A_n is a subgroup of S_n.

16) Let f be a fixed odd permutation in S_n ($n > 1$). Show that every odd permutation in S_n is a product of f and some permutation in S_n.

2 : Multiple choice Questions of 1 marks

Choose the correct option from the given options.

1) Let A , B be non empty sets and $f : A \to B$ be a permutation . Then - - -
   a) f is bijective and A = B
   b) f is one one and $A \neq B$
   c) f is bijective and $A \neq B$
   d) f is onto and $A \neq B$

2) Let A be a non empty set and $f : A \to A$ be a permutation . Then - - -
   a) f is one one but not onto
   b) f is one one and onto

c) f is onto but not one one

d) f is neither one one nor onto

3) Cycles (2 4 7) and (4 3 1) are - - -

a) inverses of each other b) disjoint

c) not disjoint d) transpositions

4) Every permutation in A_n can be written as product of - - -

a) p transpositions, where p is an odd prime

b) odd number of transpositions

c) even number of transpositions

d) none of these

5) The number of elements in S_n = - - -

a) n b) n! c) n!/2 d) 2^n

6) The number of elements in A_6 = - - -

a) 6 b) 720 c) 360 d) 2^6

7) If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 4 & 5 & 7 & 6 \end{pmatrix} \in S_7$ then α^{-1} = - - -

a) (1 2 3 6 7) b) (1 2) (3 6 7)

c) (1 2 3) (6 7) d) (4 5)

8) $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 6 & 5 & 2 \end{pmatrix} \in S_6$ is a product of - - - transpositions.

a) 1 b) 2 c) 3 d) 4

3 : Questions of 4 marks

1) Let g $\in$ S_A , A = {a_1 , a_2 , - - - , a_n}. Prove that

i) g^{-1} exists in S_A.

ii) $g\,g^{-1} = I = g^{-1}\,g$, where I is the identity permutation in S_A.

2) Let A be a non empty set with n elements. Prove that S_A is a group with respect to multiplication of permutations.

3) Let S_n be a group of permutations on n symbols $\{a_1 , a_2 , - - - , a_n\}$. prove that $o(S_n) = n!$. Also prove that S_n is not abelian if $n \geq 3$.

4) Define a cycle. Let $\alpha = (a_1 , a_2 , - - - , a_{r-1} , a_r)$ be a cycle of length r in S_n. Prove that $\alpha^{-1} = (a_r , a_{r-1} , - - - , a_2 , a_1)$.

5) Prove that every permutation in S_n can be written as a product of transpositions.

6) Prove that every permutation in S_n can be written as a product of disjoint cycles.

7) Define i) a cycle ii) a transposition. Prove that every cycle can be written as a product of transpositions.

8) Let f , g be disjoint cycles in S_A. Prove that f g = g f.

9) Define an even permutation. Express $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$ as a product of disjoint cycles. Determine whether σ is odd or even.

10) Express $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 7 & 9 & 8 & 1 & 6 \end{pmatrix}$ as a product of transpositions. State whether $\mu^{-1} \in A_9$.

11) Let $\alpha = (1\ 3\ 2\ 5)\ (1\ 4\ 3)\ (2\ 5) \in S_5$ Find α^{-1} and express it as a product of disjoint cycles. State whether $\alpha^{-1} \in A_5$.

12) Let $\lambda = (1\ 3\ 5\ 7\ 8)\ (3\ 2\ 6\ 7) \in S_8$ Find λ^{-1} and express it as a product of disjoint cycles. State whether $\lambda^{-1} \in A_8$.

13) Prove that there are exactly n!/2 even permutations and exactly n!/2 odd permutations in S_n (n>1).

14) Prove that for every subgroup H of S_n either all permutations in H are even or exactly half of them are even.

15) If f , g are even permutations in S_n then prove that f g and g^{-1} are even permutations in S_n .

16) Define an odd permutation. Let H be a subgroup of S_n, (n>1), and H contains an odd permutation. Show that o(H) is even.

17) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 4 & 9 & 7 & 8 & 3 & 2 & 1 \end{pmatrix} \in S_9$. Express α and α^{-1} as a product of disjoint cycles. State whether $\alpha^{-1} \in A_9$.

18) Let $\beta = (2\ 5\ 3\ 7)\ (4\ 8\ 2\ 1) \in S_8$. Express β as a product of disjoint cycles. State whether $\beta^{-1} \in A_8$.

19) Let G be a finite group and $a \in G$ be a fixed element. Show that f_a : $G \to G$ defined by $f_a(x) = ax$, for all $x \in G$, is a permutation on G.

20) Let G be a finite group and $a \in G$ be a fixed element. Show that f_a : $G \to G$ defined by $f_a(x) = ax^{-1}$, for all $x \in G$, is a permutation on G.

21) Let G be a finite group and $a \in G$ be a fixed element. Show that f_a : $G \to G$ defined by $f_a(x) = a^{-1}x$, for all $x \in G$, is a permutation on G.

22) Let G be a finite group and $a \in G$ be a fixed element. Show that f_a : $G \to G$ defined by $f_a(x) = axa^{-1}$, for all $x \in G$, is a permutation on G.

23) Compute $a^{-1}ba$ where $a = (2\ 3\ 5)(1\ 4\ 7)$, $b = (3\ 4\ 6\ 2) \in S_7$. Also express $a^{-1}ba$ as a product of disjoint cycles.

24) Show that there can not exist a permutation $a \in S_8$ such that $a(1\ 5\ 7)a^{-1} = (1\ 5)(2\ 4\ 6)$.

25) Show that there can not exist a permutation $a \in S_9$ such that $a(2\ 5)a^{-1} = (2\ 7\ 8)$.

26) Show that there can not exist a permutation $\mu \in S_8$ such that $\mu(1\ 2\ 6)(3\ 2)\mu^{-1} = (5\ 6\ 8)$.

27) Show that there can not exist a permutation $a \in S_7$ such that $a^{-1}(1\ 5)(2\ 4\ 6)a = (1\ 5\ 7)$.

28) Write down all permutations on 3 symbols {1, 2, 3} and prepare a composition table.

29) Show that the set of 4 permutations e = (1) , (1 2) , (3 4) , (1 2)(3 4) $\in$ S_4. form an abelian group with respect to multiplication of permutations.

30) Show that the set A = {(4) , (1 3) , (2 4) , (1 3)(2 4)} form an abelian group with respect to multiplication of permutations in S_4.

Unit – II

1 : Questions of 2 marks

1) Define i) a normal subgroup ii) a simple group.
2) Show that a subgroup H of a group G is normal if and only if $g \in G$, $x \in H \Rightarrow g^{-1}xg \in H$.
3) Show that every subgroup of an abelian group is normal.
4) Show that the alternating subgroup A_n of a symmetric group S_n is normal.
5) If a finite group G has exactly one subgroup H of a given order then show that H is normal in G.
6) Show that every group of prime order is simple.
7) Is a group of order 61 simple? Justify.
8) Define a normalizer N(H) of a subgroup H of a group G. Show that N(H) ia a subgroup of G.
9) Let H be a subgroup of a group G. Show that N(H) = G if and only if H is normal in G.
10) Define index of a subgroup. Find index of A_n in S_n, $n \geq 2$.
11) Prove that intersection of two normal subgroups of a group G is a normal subgroup of G.
12) Let H, K be normal subgroups of a group G and $H \cap K = \{e\}$. show that ab = ba for all $a \in H$, $b \in K$.

13) Prove that intersection of any finite number of normal subgroups of a group G is a normal subgroup of G.

14) Let H be a normal subgroup of a group G and K a subgroup of G such that $H \subseteq K \subseteq G$. Show that H is a normal subgroup of K.

15) Is union of two normal subgroups a normal subgroup? Justify.

16) Define a quotient group. If H is a normal subgroup of a group G then show that H is the identity element of G/H.

17) Let H be a normal subgroup of a group G and $a, b \in G$. Show that

i) $a^{-1}H = (aH)^{-1}$ ii) $(ab)^{-1}H = (bH)^{-1}(aH)^{-1}$.

18) Let $H = 3Z \subseteq (Z, +)$. Write the elements of Z/H and prepare a composition table for Z/H.

19) Let $H = 4Z \subseteq (Z, +)$. Write the elements of Z/H and prepare a composition table for Z/H.

20) Prove that the quotient group of an abelian group is abelian.

21) Give an example of an abelian group G/H such that G is not abelian. Explain.

22) Give an example of a cyclic group G/H such that G is not cyclic. Explain.

23) Let H, K be normal subgroups of a group G. If G/H = G/K then show that H = K.

24) Let H be a normal subgroup of a group G. If G/H is abelian then show that $xyx^{-1}y^{-1} \in G$, for all $x, y \in G$.

25) Let H be a normal subgroup of a group G. If $xyx^{-1}y^{-1} \in H$, for all $x, y \in G$ then show that G/H is abelian.

26) If H is a normal subgroup of a group G and $i_G(H) = m$ then show that $x^m \in H$, for all $x \in G$.

27) Show that every subgroup of a cyclic group is normal.

28) Give an example of a non cyclic group in which every subgroup is normal.

29) If H is a subgroup of a group G and N a normal subgroup of G then show that $H \cap N$ is a normal subgroup of H.

30) If H, K are normal subgroups of a group G then show that a subgroup HK is normal in G.

31) Let H be a subgroup of index 2 of a group G. If $x \in G$ then show that $x^2 \in H$.

2 : Multiple choice Questions of 1 marks

Choose the correct option from the given options.

1) The number of normal subgroups in a nontrivial simple group = - - -

   a) 0 b) 1 c) 2 d) 3

2) In any abelian group every subgroup is - - -

   a) cyclic b) normal c) finite d) $\{e\}$

3) Order of a group $Z/3Z$ = - - -

   a) 0 b) 1 c) 3 d) ∞

4) A proper subgroup of index - - - is always normal.

   a) 1 b) 2 c) 3 d) 6

5) Let H be a normal subgroup of order 2 in a group G. Then - - -

   a) $H = G$ b) $H \subseteq Z(G)$

   c) $Z(G) \subseteq H$ d) neither $H \subseteq Z(G)$ nor $Z(G) \subseteq H$

6) For a group G, the center $Z(G)$ is defined as - - -

   a) $\{x \in G : ax = xa, \text{ for all } a \in G\}$

   b) $\{x \in G : ax = xa, \text{ for some } a \in G\}$

   c) $\{x \in G : x^2 = x\}$

   d) $\{x \in G : x^2 = e\}$

7) Every subgroup of a cyclic group is - - -

a) cyclic and normal b) cyclic but not normal

c) normal but not cyclic d) neither cyclic nor normal

8) Index of A_3 in S_3 is - - -

a) 1 b) 2 c) 3 d) 6

3 : Questions of 3 marks

1) Define center of a group. Show that center of a group is a normal subgroup.
2) Show that a normal subgroup of order 2 in a group G is contained in the center of G.
3) Prove that a subgroup H of a group G is normal if and only if $gHg^{-1} = H$, for all $g \in G$.
4) Let H , K be subgroups of a group G. If H is normal then show that HK is a subgroup of G.
5) Let H , K be subgroups of a group G. If K is normal then show that HK is a subgroup of G.
6) If H is a subgroup of a group G then show that N(H) is the largest subgroup of G in which H is normal.
7) Prove that a non empty subset H of a group G is normal subgroup of G if and only if $x , y \in H , g \in G \Rightarrow (gx)(gy)^{-1} \in H$.
8) Prove that a subgroup H of a group G is normal if and only if $Hx = xH$, for all $x \in G$.
9) Prove that a subgroup H of a group G is normal if and only if $HaHb = Hab$, for all $a , b \in G$.
10) Prove that a subgroup H of a group G is normal if and only if $aHbH = abH$, for all $a , b \in G$.

11) Let H be a subgroup of a group G. If product of any two right cosets of H in G is again a right coset of H in G then prove that H is normal.

12) Let H be a subgroup of a group G. If product of any two left cosets of H in G is again a left coset of H in G then prove that H is normal.

13) Define index of a subgroup. Show that any subgroup of index 2 is normal.

14) Define a group of quarterions and find all its normal subgroups.

15) If a cyclic subgroup of T of a group G is normal in G then show that every subgroup of T is normal in G.

16) Let H be a normal subgroup of a group G. Show that $\cap \{xHx^{-1} : x \in G\}$ is a normal subgroup of G.

17) Let H , K be normal subgroups of a group G. If o(H) , o(K) are relatively prime numbers then show that xy = yx, for all $x \in H$, $y \in K$.

18) Let H , K be normal subgroups of a group G. If $H \cup K$ is a normal subgroup of G then show that $H \subseteq K$ or $K \subseteq H$

19) Let H_1 , H_2 , - - - , H_n be proper normal subgroups of a group G such that $G = \bigcup_{i=1}^{n} H_i$ and $H_i \cap H_j = \{e\}$, for all $i \neq j$. Prove that G is an abelian group.

20) Write the elements of S_3 and A_3 on three symbols {1, 2, 3}. Prepare a composition table for S_3/A_3.

21) Prove that the quotient group of a cyclic group is cyclic.

22) Let H be a normal subgroup of a finite group G and o(H) , $i_G(H)$ are relatively prime numbers If $x \in G$ and $x^{o(H)} = e$ then show that $x \in H$.

23) Let H be a subgroup of a group G. Prove that $xHx^{-1} = H$, for all $x \in G$ if and only if Hxy = HxHy , for all x , $y \in G$.

24) Let H be a subgroup of a group G. Prove that $xHx^{-1} = H$, for all $x \in G$ if and only if $xyH = xHyH$, for all $x , y \in G$.

25) Show that a subgroup H of a group G is normal if and only if $xy \in H \Rightarrow yx \in H$, where $x , y \in G$.

26) Show that a subgroup H of a group G is normal if and only if the set $\{Hx : x \in G\}$ of all right cosets of H in G is closed under multiplication.

27) Let H be a subgroup of a group G and $x^2 \in H$, for all $x \in G$. Show that H is normal in G

28) Let G be a group and $a \in G$. Denote $N(a) = \{x \in G : xa = ax\}$ Show that $a \in Z(G)$ if and only if $N(a) = G$.

29) Let N be a normal subgroup of a group G and H a subgroup of G. If o(G/N) and o(H) are relatively prime numbers then show that $H \subseteq N$.

30) Write any six equivalent conditions of normal subgroup.

Unit – III

1 : Questions of 2 marks

1) Let (R , +) be a group of real numbers under addition. Show that $f : R \to R$, defined by $f(x) = 3x$, for all $x \in R$, is a group homomorphism. Find Ker(f).

2) Let (R , +) be a group of real numbers under addition. Show that $f : R \to R$, defined by $f(x) = 2x$, for all $x \in R$, is a group homomorphism. Find Ker(f).

3) If (R , +) is a group of real numbers under addition and $(R^+ , \cdot)$ is a group of positive real numbers under multiplication. Show that $f : R \to R^+$, defined by $f(x) = e^x$, for all $x \in R$, is a group homomorphism. Find Ker(f).

4) Let $(R^* , \cdot)$ be a group of non zero real numbers under multiplication. Show that $f : R^* \to R^*$, defined by $f(x) = x^3$, for all $x \in R^*$, is a group homomorphism. Find Ker(f).

5) Let $(C^*, \cdot)$ be a group of non zero complex numbers under multiplication. Show that $f : C^* \to C^*$, defined by $f(z) = z^4$, for all $z \in C^*$, is a group homomorphism. Find Ker(f).

6) Let $(Z, +)$ be a group of integers under addition and $G = \{5^n : n \in Z\}$ a group under multiplication. Show that $f : Z \to G$, defined by $f(n) = 5^n$, for all $n \in Z$, is onto group homomorphism.

7) Let $(Z, +)$ and $(E, +)$ be the groups of integers and even integers respectively under addition. Show that $f : Z \to E$, defined by $f(n) = 2n$, for all $n \in Z$, is an isomorphism.

8) Define a group homomorphism. Let $(G, *)$, $(G', *')$ be groups with identity elements e, e' respectively. Show that $f : G \to G'$, defined by $f(x) = e'$, for all $x \in G$, is a group homomorphism.

9) Let $G = \{a, a^2, a^3, a^4, a^5 = e\}$ be the cyclic group generated by a. Show that $f : (Z_5, +_5) \to G$, defined by $f(\bar{n}) = a^n$, for all $\bar{n} \in Z_5$, is a group homomorphism. Find Ker(f).

10) Let $f : (R, +) \to (R, +)$ be defined by $f(x) = x + 1$, for all $x \in R$. Is f a group homomorphism? Why?

11) Let $G = \{1, -1, i, -i\}$ be a group under multiplication and $Z'_8 = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ a group under multiplication modulo 8. Show that G and Z'_8 are not isomorphic.

12) Show that the group $(Z_4, +_4)$ is isomorphic to the group $(Z'_5, \times_5)$.

13) Let $f : G \to G'$ be a group homomorphism. If $a \in G$ and $o(a)$ is finite then show that $o(f(a)) \mid o(a)$.

14) Let $f : G \to G'$ be a group homomorphism If H' is a subgroup of G' then show that $\text{Ker}(f) \subseteq f^{-1}(H')$.

15) Let $f : G \to G'$ be a group homomorphism and $o(a)$ is finite, for all $a \in G$. If f is one one then show that $o(f(a)) = o(a)$.

16) Let $f : G \to G'$ be a group homomorphism and $o(f(a)) = o(a)$, for all $a \in G$. Show that f is one one.

2 : Multiple choice Questions of 1 marks

Choose the correct option from the given options.

1) Every finite cyclic group of order n is isomorphic to - - -
   a) $(Z , +)$ b) $(Z_n , +_n)$ c) $(Z_n , \times_n)$ d) $(Z'_n , \times_n)$
2) Every infinite cyclic group is isomorphic to - - -
   a) $(Z , +)$ b) $(Z_n , +_n)$ c) $(Z_n , \times_n)$ d) $(Z'_n , \times_n)$
3) Let $f : G \to G'$ be a group homomorphism and $a \in G$. If $o(a)$ is finite then - - -
   a) $o(f(a)) = \infty$ b) $o(f(a)) \mid o(a)$.
   c) $o(a) \mid o(f(a))$ d) $o(f(a)) = 0$.
4) A group $G = \{1 , -1 , i , -i\}$ under multiplication is not isomorphic to - - -
   a) $(Z_4 , +_4)$ b) G
   c) $(Z'_8 , \times_8)$ d) none of these.
5) Let $f : G \to G'$ be a group homomorphism. If G is abelian then f(G) is - - -
   a) non abelian b) abelian
   c) cyclic d) empty set
6) Let $f : G \to G'$ be a group homomorphism. If G is cyclic then f(G) is - - -
   a) non abelian b) non cyclic
   c) cyclic d) finite set
7) A onto group homomorphism $f : G \to G'$ is an isomorphism if Ker(f) = - - -
   a) ϕ b) $\{e\}$ c) $\{e'\}$ d) none of these

8) A function $f : G \to G$, (G is a group) , defined by $f(x) = x^{-1}$, for all $x \in G$, is an automorphism if and only if G is - - -

a) abelian b) cyclic c) non abelian d) $G = \phi$.

3 : Questions of 4 marks

1) Let $f : G \to G'$ be a group homomorphism . prove that f(G) is a subgroup of G'. Also prove that if G is abelian then f(G) is abelian.
2) Let $f : G \to G'$ be a group homomorphism. Show that f is one one if and only if $Ker(f) = \{e\}$.
3) Let $G = \{1 , -1 , i , -i\}$ be a group under multiplication. Show that $f : (Z , +) \to G$, defined by $f(n) = i^n$, for all $n \in Z$, is onto group homomorphism. Find Ker(f).
4) Let $G = \{1 , -1 , i , -i\}$ be a group under multiplication. Show that $f : (Z , +) \to G$, defined by $f(n) = (-i)^n$, for all $n \in Z$, is onto group homomorphism. Find Ker(f).
5) Let $G = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in R,\ a^2 + b^2 \neq 0 \right\}$ be a group under multiplication and C^* be a group of non zero complex numbers under multiplication. Show that $f : C^* \to G$ defined by $f(a + ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, for all $a + ib \in C^*$, is an isomorphism.
6) Define a group homomorphism. Prove that homomorphic image of a cyclic group is cyclic.
7) Let $f : G \to G'$ be a group homomorphism. Prove that
   - i) f(e) is the identity element of G', where e is the identity element of G
   - ii) $f(a^{-1}) = (f(a))^{-1}$, for all $a \in G$
   - iii) $f(a^m) = (f(a))^m$, for all $a \in G$, $m \in Z$.

8) Let $(C^*, \cdot)$.$(R^*, \cdot)$ be groups of non zero complex numbers, non zero real numbers respectively under multiplication. Show that $f : C^* \to R^*$ defined by $f(z) = |z|$, for all $z \in C^*$, is a group homomorphism. Find Ker(f). Is f onto? Why?

9) Let $(C^*, \cdot)$, $(R^*, \cdot)$ be groups of non zero complex numbers, non zero real numbers respectively under multiplication. Show that $f : C^* \to R^*$ defined by $f(z) = |\bar{z}|$, for all $z \in C^*$, is a group homomorphism. Find Ker(f). Is f onto? Why?

10)Let $G = \{1, -1\}$ be a group under multiplication. Show that $f : (Z, +) \to G$ defined by $f(n) = \begin{cases} 1 & , \text{ if n is even} \\ -1 & , \text{ if n is odd} \end{cases}$

is onto group homomorphism. Find Ker(f).

11)Let $(R^+, \cdot)$ be a group of positive reals under multiplication. Show that $f : (R, +) \to R^+$ defined by $f(x) = 2^x$, for all $x \in R$, is an isomorphism.

12) Let $(R^+, \cdot)$ be a group of positive reals under multiplication. Show that $f : (R, +) \to R^+$ defined by $f(x) = e^x$, for all $x \in R$, is an isomorphism.

13) If $f : G \to G'$ is an isomorphism and $a \in G$ then show that $o(a) = o(f(a))$.

14) Prove that every finite cyclic group of order n is isomorphic to $(Z_n, +_n)$.

15)Prove that every infinite cyclic group is isomorphic to $(Z, +)$.

16)Let G be a group of all non singular matrices of order 2 over the set of reals and R^* be a group of all nonzero reals under multiplication. Show that $f : G \to R^*$, defined by $f(A) = |A|$, for all $A \in G$, is onto group homomorphism. Is f one one? Why?

17) Let G be a group of all non singular matrices of order n over the set of reals and R^* be a group of all nonzero reals under multiplication. Show that $f : G \to R^*$, defined by $f(A) = |A|$, for all $A \in G$, is onto group homomorphism.

18) Let R^* be a group of all nonzero reals under multiplication. Show that $f : R^* \rightarrow R^*$, defined by $f(x) = |x|$, for all $x \in R^*$, is a group homomorphism. Is f onto? Justify.

19) Prove that every group is isomorphic to it self. If G_1, G_2 are groups such that $G_1 \cong G_2$ then prove that $G_2 \cong G_1$.

20) Let G_1, G_2, G_3 be groups such that $G_1 \cong G_2$ and $G_2 \cong G_3$. Prove that $G_1 \cong G_3$.

21) Show that $f : (C , +) \rightarrow (C , +)$ defined by $f(a + ib) = -a + ib$, for all $a + ib \in C$, is an automorphism.

22) Show that $f : (C , +) \rightarrow (C , +)$ defined by $f(a + ib) = a - ib$, for all $a + ib \in C$, is an automorphism.

23) Show that $f : (Z , +) \rightarrow (Z , +)$ defined by $f(x) = -x$, for all $x \in Z$, is an automorphism.

24) Let G be an abelian group. Show that $f : G \rightarrow G$ defined by $f(x) = x^{-1}$, for all $x \in G$, is an automorphism.

25) Let G be a group and $a \in G$. Show that $f_a : G \rightarrow G$ defined by $f_a(x) = axa^{-1}$, for all $x \in G$, is an automorphism.

26) Let G be a group and $a \in G$. Show that $f_a : G \rightarrow G$ defined by $f_a(x) = a^{-1}xa$, for all $x \in G$, is an automorphism.

27) Let $G = \{a , a^2 , a^3 , --- , a^{12} (= e)\}$ be a cyclic group generated by a. Show that $f : G \rightarrow G$ defined by $f(x) = x^4$, for all $x \in G$, is a group homomorphism. Find Ker(f).

28) Let $G = \{a , a^2 , a^3 , --- , a^{12} (= e)\}$ be a cyclic group generated by a. Show that $f : G \rightarrow G$ defined by $f(x) = x^3$, for all $x \in G$, is a group homomorphism. Find Ker(f).

29) Show that $f : (C , +) \rightarrow (R , +)$ defined by $f(a + ib) = a$, for all $a + ib \in C$, is onto homomorphism. Find Ker(f).

30) Show that homomorphic image of a finite group is finite. Is the converse true? Justify.

Unit – IV

1 : Questions of 2 marks

1) In a ring $(Z , \oplus , \odot)$, where $a \oplus b = a + b - 1$ and $a \odot b = a + b - ab$, for all $a , b \in Z$, find zero element and identity element.
2) Define an unit. Find all units in $(Z_6 , +_6 , \times_6)$.
3) Define a zero divisor. Find all zero divisors in $(Z_8 , +_8 , \times_8)$.
4) Let R be a ring with identity 1 and $a \in R$. Show that

   i) $(-1)a = -a$ ii) $(-1)(-1) = 1$
5) Let R be a commutative ring and $a , b \in R$. Show that $(a - b)^2 = a^2 - 2ab + b^2$.
6) Let $(Z[\sqrt{-5}] , + , \cdot)$ be a ring under usual addition and multiplication of elements of $Z[\sqrt{-5}]$. Show that $Z[\sqrt{-5}]$ is a commutative ring . Is $2 + 3\sqrt{-5}$ a unit in $Z[\sqrt{-5}]$?
7) Let $\overline{m} \in (Z_n , +_n , \times_n)$ be a zero divisor. Show that m is not relatively prime to n, where $n > 1$.
8) If $\overline{m} \in (Z_n , +_n , \times_n)$ is invertible then show that m and n are relatively prime to n, where $n > 1$.
9) Let $n > 1$ and $0 < m < n$. If m is relatively prime to n then show that $\overline{m} \in (Z_n , +_n , \times_n)$ is invertible.
10) Let $n > 1$ and $0 < m < n$. If m is not relatively prime to n then show that $\overline{m} \in (Z_n , +_n , \times_n)$ is a zero divisor.
11) Show that a field has no zero divisors.
12) Let R be a ring in which $a^2 = a$, for all $a \in R$. Show that $a + a = 0$, for all $a \in R$.

13) Let R be a ring in which $a^2 = a$, for all $a \in R$. If $a, b \in R$ and $a + b = 0$, then show that $a = b$.

14) Let R be a commutative ring with identity 1. If a , b are units in R then show that a^{-1} and ab are units in R.

15) In $(Z_{12}, +_{12}, \times_{12})$ find (i) $(\bar{3})^2 +_{12} (\bar{5})^{-2}$ (ii) $(\bar{7})^{-1} +_{12} \bar{8}$.

16) In $(Z_{12}, +_{12}, \times_{12})$ find (i) $(\bar{5})^{-1} - \bar{7}$ (ii) $(\overline{11})^{-2} +_{12} \bar{5}$.

2 : Multiple choice Questions of 1 marks

Choose the correct option from the given options.

1) $R = \{\pm 1, \pm 2, \pm 3, - - -\}$ is not a ring under usual addition and multiplication of integers because - - -
   a) R is not closed under multiplication
   b) R is not closed under addition
   c) R does not satisfy associativity w.r.t. addition
   d) R does not satisfy associativity w.r.tmultiplication

2) Number of zero divisors in $(Z_6, +_6, \times_6)$ = - - -
   a) 0 b) 1 c) 2 d) 3

3) $(Z_{43}, +_{43}, \times_{43})$ is - - -
   a) both field and integral domain
   b) an integral domain but not a field
   c) a field but not an integral domain
   d) neither a field nor an integral domain

4) In $(Z_9, +_9, \times_9)$, $\bar{6}$ is - - -
   a) a zero divisor b) an invertible element
   c) a zero element d) an identity element

5) Every Boolean ring is - - -
   a) an integral domain b) a field
   c) a commutative ring d) a division ring

6) If a is a unit in a ring R then a is - - -

   a) a zero divisor   b) an identity element

   c) a zero element   d) an invertible element

7) If R is a Boolean ring and $a \in R$ then - - -

   a) $a + a = a$ b) $a^2 = 0$ c) $a^2 = 1$ d) $a + a = 0$

8) Value of $(\bar{7})^2 - \bar{7}$ in $(Z_8, +_8, \times_8)$ is - - -

   a) $\bar{6}$ b) $\bar{4}$ c) $\bar{2}$ d) $\bar{0}$

3 : Questions of 6 marks

1) a) Define i) a ring ii) an integral domain iii) a division ring.

   b) Show that the set R = {0, 2, 4, 6} is a commutative ring under addition and multiplication modulo 8.

2) a) Define i) a commutative ring ii) a field iii) a skew field.

   b) In 2Z, the set of even integers, we define a + b = usual addition of a and b and $a \odot b = \frac{ab}{2}$. Show that $(2Z, +, \odot)$ is a ring.

3) a) Define i) a ring with identity element ii) an unit element iii) a Boolean ring.

   b) Let (2Z , +) be an abelian group of even integers under usual addition. Show that $(2Z, +, \odot)$ is a commutative ring with identity 2, where $a \odot b = \frac{ab}{2}$, for all $a, b \in 2Z$.

4) a) Define i) a zero divisor ii) an invertible element iii) a field.

   b) Let (3Z , +) be an abelian group under usual addition where $3Z = \{3n \mid n \in Z\}$. Show that $(3Z, +, \odot)$ is a commutative ring with identity 3, where $a \odot b = \frac{ab}{3}$, for all $a, b \in 3Z$.

5) a) Let $(R, +, \cdot)$ be a ring and $a, b, c \in R$. Prove that

i) $a \cdot 0 = 0$ ii) $(a - b)c = ac - bc$.

b) Show that $(Z , \oplus , \odot)$is a ring, where $a \oplus b = a + b - 1$ and $a \odot b = a + b - ab$, for all $a , b \in Z$.

6) a) Let $(R, +, \cdot)$ be a ring and $a, b, c \in R$. Prove that

i) $a \cdot (-b) = -(ab)$ ii) $a (b - c)c = ab - ac$.

b) Show that the abelian group $(Z[\sqrt{-5}] , +)$ is a ring under multiplication

$(a + b\sqrt{-5})(c + d\sqrt{-5}) = ac-5bd + (ad + bc) \sqrt{-5}$.

7) a) Define i) a division ring ii) an unit element iii) an integral domain

b) Show that the abelian group $(Z[i] , +)$ is a ring under multiplication

$(a + bi)(c + di) = ac - bd + (ad + bc) i$, for all $a + bi ,c + di \in Z[i]$.

8) a) Let R be a ring with identity 1 and $(ab)^2 = a^2b^2$, for all $a, b \in R$. Show that R is commutative.

b) Show that the abelian group $(Z_n , +_n)$ is a commutative ring with identity $\bar{1}$ under multiplication modulo n operation.

9) a) Show that a ring R is commutative if and only if $(a + b)^2 = a^2 + 2ab + b^2$, for all $a, b \in R$.

b) Show that $Z[i] = \{a + ib \mid a , b \in Z\}$, the ring of Gaussian integers, is an integral domain.

10)a) Show that a commutative ring R is an integral domain if and only if $a , b , c \in R$, $a \neq 0$, $ab = ac \Rightarrow b = c$.

b) Prepare addition modulo 4 and multiplication modulo 4 tables. Find all invertible elements in Z_4.

11) a) Show that a commutative ring R is an integral domain if and only if $a , b \in R$, $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$.

b) Prepare addition modulo 5 and multiplication modulo 5 tables. Find all invertible elements in Z_5.

12)a) Let R be a commutative ring. Show that the cancellation law with respect to multiplication holds in R if and only if $a, b \in R$, $ab = 0$ $\Rightarrow$ either $a = 0$ or $b = 0$.

b) Prepare a multiplication modulo 6 table for a ring $(Z_6, +_6, \times_6)$. Hence find all zero divisors and invertible elements in Z_6.

13 a) For $n > 1$, show that Z_n is an integral if and only if n is prime.

b) Let $R = \left\{ \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} : z, w \in C \right\}$ be a ring under addition and multiplication, where $C = \{a + ib \mid a, b \in R\}$. Show that R is a divison ring.

14 a) Prove that every field is an integral domain. Is the converse true? Justify.

b) Which of the following rings are fields? Why?

i) $(Z, +, \times)$ ii) $(Z_5, +_5, \times_5)$ iii) $(Z_{25}, +_{25}, \times_{25})$.

15) a) Prove that every finite integral domain is a field.

b) Which of the following rings are integral domains? Why?

i) $(2Z, +, \times)$ ii) $(Z_{50}, +_{50}, \times_{50})$ iii) $(Z_{17}, +_{17}, \times_{17})$.

16 a) Prove that a Boolean ring is a commutative ring.

b) Give an example of a division ring which is not a field.

17 a) for $n > 1$, show that Z_n is a field if and onle if n is prime.

b) Let $R = \{a + bi + cj + dk \mid a, b, c, d \in R\}$, where $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. Show that every nonzero element of R is invertible.

18 a) If R is a ring and $a, b \in R$ then prove or disprove $(a + b)^2 = a^2 + 2ab + b^2$.

b) Show that R^+, the set of all positive reals forms a ring under the following binary operations :

$a \oplus b = ab$ and $a \odot b = a^{\log_5 b}$, for all $a,b \in R^+$.

19) a) Define i) a ring ii) a Boolean ring iii) an invertible element.

b) Let p be a prime and $(pZ , +)$ be an abelian group under usual addition, show that $(pZ, + , \odot)$ is a commutative ring with identity element p where $a \odot b = \frac{ab}{p}$, for all $a , b \in pZ$.

20) a) Define i) a ring with identity element ii) a commutative ring iii) a zero divisor.

b) Show that R^+ , the set of all positive reals forms a ring under the following binary operations :

$a \oplus b = ab$ and $a \odot b = a^{\log_7 b}$, for all $a,b \in R^+$.

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